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The Validity of Nonlinear Geometric Optics for Weak Solutions of Conservation Laws

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Abstract. The method of weakly nonlinear geometric optics is one of the main formal perturbation techniques used in analyzing nonlinear wave motion for hyperbolic systems. The tacit assumption in using such perturbation methods is that the corresponding solutions of the hyperbolic system remain smooth; since shock waves typically form in such solutions, these assumptions are rarely satisfied in practice. Nevertheless, in a variety of applied contexts, these methods give qualitatively reliable answers for discontinuous weak solutions. Here we give a rigorous proof for the validity of nonlinear geometric optics for general weak solutions of systems of hyperbolic conservation laws in a single space variable. The methods of proof do not mimic the formal construction of weakly nonlinear asymptotics but instead rely on structural symmetries of the approximating equations, stability estimates for intermediate asymptotic times, and the rapid decay in variation of weak solutions for large asymptotic times.

1. Introduction

After the early work of Landau [13], Lighthill [9], and Whitham [26], the method of weakly nonlinear geometric optics has evolved into one of the main perturbation techniques for analyzing solutions of quasi-linear hyperbolic equations (see [24] for a review of the applied literature before 1981). There are recent multi-dimensional applications of these methods to the development of simplified models in reacting gas flow [23], to the regular reflection of weak shocks [8], and to the formation of Mach stems in reacting shock fronts [22]. Furthermore, systematic self-consistent derivations using nonlinear geometric optics for simplified theories of both nonresonant and resonant wave interactions in several space dimensions have been developed recently ([9, 10, 20]). While the tacit assumptions used in the formal derivation of the expansion of weakly nonlinear geometric optics are that the

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