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Quantum Measures and States on Jordan Algebras

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Abstract. A problem of Mackey for von Neumann algebras has been settled by the conjunction of the early work of Gleason and the recent advances of Christensen and Yeadon. We show that Mackey's conjecture holds in much greater generality. Let A be a JBW-algebra and let L be the lattice of all projections in A. A quantum measure on L is a countably additive map, m, from L into the real numbers. Our results imply that m always has a unique extension to a bounded linear functional on A, provided that A has no Type I_2 direct summand.

Introduction

Let W be a von Neumann algebra or a JBW-algebra (see below for definitions). Let $P(W)$ be the lattice of all projections in W. A *measure* on $P(W)$ is a positive, real-valued function, μ, on $P(W)$ such that $\mu(0)=0$ and, whenever p and q are orthogonal projections, $\mu(p+q)=\mu(p)+\mu(q)$. If, whenever (p_i) is a countable family of orthogonal projections in W, $\mu(\sum p_i)=\sum \mu(p_i)$, then μ is said to be *countably additive*. Clearly, each positive linear functional on W restricts to a measure on $P(W)$. When W is the algebra of complex two-by-two matrices, or a spin factor, there exist measures on $P(W)$ which do not correspond to linear functionals on W.

Over twenty-five years ago, Mackey conjectured:

When W is a von Neumann algebra with no Type I_2 direct summand and μ is any countably additive probability measure on $P(W)$ then μ can be extended to a state of W.

Very recently the problem of establishing Mackey's conjecture has been completely solved by Christensen [7] and Yeadon [33, 34]. Christensen used great ingenuity and insight to solve the problem for properly infinite von Neumann and for von Neumann algebras of Type I_n, where $3 \leq n < \infty$. Yeadon devised different methods to deal with general finite von Neumann algebras and so complete the solution. We are grateful to the referee for drawing our attention to the work of