Commun. Math. Phys. 97, 461–463 (1985)

Communications in Mathematical Physics

A Remark on the Cluster Theorem

K. Fredenhagen*
Centre de Physique Théorique,** CNRS Luminy Case 907, F-13288 Marseille Cedex 9, France

Abstract. An improved version of the cluster theorem of relativistic quantum field theory with the correct exponential decay is derived by function theoretical methods.

In its simplest form the cluster theorem of relativistic quantum field theory states that in a massive theory the correlations of local observables in the vacuum state decrease exponentially with their spacelike separation. The standard proof of this fact [1] uses the method of the Jost–Lehmann–Dyson representation and leads to the estimate

$$|(\Omega, AB\Omega) - (\Omega, A\Omega)(\Omega, B\Omega)| \leqq c(\tau)e^{-m\tau}\{\|A^*\Omega\| \, \|HB\Omega\| + \|HA^*\Omega\| \, \|B\Omega\|\}, \quad (1)$$

where H is the Hamiltonian, m the mass gap, Ω the vacuum vector, A and B are local operators and τ is the spacelike distance of the origin from the spacelike complement of a wedge[1] W with the property that $[\alpha_x(A), B] = 0$ for $x \in W$, α_x denoting the translation of an observable by x. The function $c(\tau)$ depends on the volume of the spatial localization regions of A and B and is polynomially decreasing for finite volumes and at most polynomial increasing for unbounded localization regions (e.g. wedges).

This theorem gives no information on the correlations of local operators for more complicated geometrical situations, for instance if A is localized in a double cone O_1 and B is known to commute with all operators in a larger double cone O_2. A further unsatisfying feature of the estimate (1) is the occurrence of the Hamiltonian in the bound on the right-hand side. The first proof of the cluster theorem by Ruelle [2] avoids these drawbacks; however, it apparently leads only to a decay faster than any power of the distance,

$$|(\Omega, AB\Omega) - (\Omega, A\Omega)(\Omega, B\Omega)| \leqq c_N \tau^{-N}\{\|A^*\Omega\| \, \|B\Omega\| + \|A\Omega\| \, \|B^*\Omega\|\}, \quad (2)$$

where $\tau > 0$ such that $[\alpha_t(A), B] = 0$, $|t| \leqq \tau$, α_t denoting the time translations, and

---

* Heisenberg fellow

** Laboratoire propre du Centre National de la Recherche Scientifique

1 A wedge is the image of the set $\{|x^0| < x^1\}$ under a Poincaré transformation