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# Universal Lower Bounds on Eigenvalue Splittings for One Dimensional Schrödinger Operators

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**Abstract.** We provide lower bounds on the eigenvalue splitting for $-d^2/dx^2 + V(x)$ depending only on qualitative properties of $V$. For example, if $V$ is $C^\infty$ on $[a,b]$ and $E_n, E_{n-1}$ are two successive eigenvalues of $-d^2/dx^2 + V$ with $u(a) = u(b) = 0$ boundary conditions, and if $\lambda = \max_{E\in(E_{n-1},E_n);x\in(a,b)} |E - V(x)|^{1/2}$, then

$$E_n - E_{n-1} \geqq \pi\lambda^2 \exp[-\lambda(b-a)].$$

The exponential factor in such bounds are saturated precisely in tunneling examples. Our results are *not* restricted to $V$'s of compact support, but only require $E_n < \varliminf_{x\to\infty} V(x)$.

## 1. Introduction

There are two cases where it is well known that Schrödinger operators have non-degenerate eigenvalues: The lowest eigenvalue in general dimension and all one dimensional eigenvalues. One can ask about making this quantitative, i.e. obtain explicit lower bounds on the distance to the nearest eigenvalues. Obviously, one cannot hope to do this without any restriction on $V$, since, for example, if $\chi$ is the characteristic function of $(-1,1)$, one can show that, for $\ell$ large, $-d^2/dx^2 - \chi(x) - \chi(x-\ell)$ has at least two eigenvalues and $E_1 - E_0 \to 0$ as $\ell \to \infty$ (see e.g. Harrell [5]). Thus, we ask the following: Can one obtain lower bounds on eigenvalues splittings only in terms of geometric properties of the set where $V(x) < E$ ($E$ at or near the eigenvalues in question) and the size of $V$ on this set. We will do precisely this for the one dimensional case in this note, and we will prove results on the ground state in multi-dimensions in [8] (see also Wong, Yau and Yau [12]).

While these questions are of interest for their own value, we came upon them with specific applications in mind [7, 9].

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