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A Proof of the Axial Anomaly

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Abstract. The local form of the axial anomaly with both left and right-handed gauge fields and a metric present is given and proved using the families index theorem

The axial anomaly is the variation of the determinant of the Dirac operator $\not{D}$ under axial changes of the gauge field [1]. More specifically, the zeta-function definition of operator determinants gives that $\ln\det\not{D} = -\frac{1}{2}$(finite part at $s=0$) $(d/ds)\operatorname{Tr}(\not{D}^2)^{-s}$. Under a variation $\delta\not{D} = \{\alpha\gamma^5, \not{D}\}$, $\delta\ln\det\not{D} = 2\lim_{s\to 0}\operatorname{Tr}\alpha\gamma^5(\not{D}^2)^{-s}$. Thus all we need know is $\lim_{s\to 0}\operatorname{Tr}\gamma^5(\not{D}^2)^{-s}(x,x) = \lim_{s\to 0}(1/\Gamma(s))\int_0^\infty \operatorname{Tr}\gamma^5 T^{s-1}\cdot e^{-T\not{D}^2}(x,x)\,dT$, which is given by a certain term in the asymptotic expansion of $e^{-T\not{D}^2}(x,x)$. It has become clear that the nontriviality of this variation is related to the topology of the space of gauge fields modulo gauge transformations [2]. We wish to show that in fact the exact form of the anomaly is given from topological arguments. In the physics literature the above heat kernel term is computed by expanding the kernel perturbatively around the flat kernel [1]. We believe that the following is the first nonperturbative (i.e. nondiagrammatic) proof of the axial anomaly in arbitrary dimension. The situation is similar to the special case of vector gauge fields, in which the heat kernel expansion of the square of the Dirac operator can be used to prove the index theorem [3]. With axial gauge fields present the direct analysis is surprisingly complicated. We work backwards and use the families index theorem along with invariance arguments to derive the expression for the local anomaly.

In the physics terminology, Lemma 2 below is the Wess-Zumino consistency condition and Lemma 4 amounts to showing the uniqueness of its nontrival solution.

Let A^+, A^- be connections on a principal G-bundle $G\to P\to M$ with G a compact Lie group and M an even n-dimensional closed Riemannian spin manifold with metric g. Let V be an associated bundle to P and let $S = S^+\oplus S^-$ be the

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