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Intersection Properties of Simple Random Walks: A Renormalization Group Approach

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Abstract. We study estimates for the intersection probability, $g(m)$, of two simple random walks on lattices of dimension $d=4$, $4-\varepsilon$ as a problem in Euclidean field theory. We rigorously establish a renormalization group flow equation for $g(m)$ and bounds on the β-function which show that, in $d=4$, $g(m)$ tends to zero logarithmically as the killing rate (mass) m tends to zero, and that the fixed point, g^*, in $d=4-\varepsilon$ is bounded by $\text{const}'\varepsilon \leqq g^* \leqq \text{const}\,\varepsilon$. Our methods also yield estimates on the intersection probability of three random walks in $d=3$, $3-\varepsilon$. For $\varepsilon=0$, these results were first obtained by Lawler [1].

1. Introduction

Two Brownian paths in $\mathbb{R}^d$ starting at different points intersect with positive probability in less than four dimensions, but do never intersect in four or more dimensions [2, 3].

The continuum limit of $g_0|\boldsymbol{\phi}|_d^4$ theory, $\boldsymbol{\phi}=\phi$, or $\boldsymbol{\phi}=(\phi^1,\phi^2)$, $g_0>0$, is an interacting theory in less than four dimensions, in the superrenormalizable regime [4], but is a (generalized) free field in more than four dimensions [5]. Results in four dimensions remain incomplete, but there are strong reasons to expect that the continuum limit is trivial in that case, too.

Symanzik recognized the connection between these two facts in his work [6] on a representation of $g_0|\boldsymbol{\phi}|^4$-theory as a gas of Brownian paths with local, repulsive interaction. Further work on that connection led to a novel, rather intuitive approach to scalar quantum field theory to which several people contributed valuable results, in the past few years. (See e.g. [7] and references therein for reviews of recent results.)

On a more abstract, more heuristic level, much insight into the theory of critical points in lattice field theories and the related problem of constructing continuum limits in dimensions close to four has been accomplished by using renormalization group methods; see e.g. [8] and references given there. In particular, for $g_0|\boldsymbol{\phi}|^4$-theories, perturbative renormalization group equations predict that, in four dimensions, the renormalized coupling constant $g=g(m)$ tends to zero like