Commun. Math. Phys. 97, 91–110 (1985)

Communications in
Mathematical
Physics

The Intersection of Brownian Paths as a Case Study of a Renormalization Group Method for Quantum Field Theory

Michael Aizenman*

Departments of Mathematics and Physics, Rutgers University, New Brunswick, NJ 08903, USA

Abstract. A new approach is presented for the study of the probability that the random paths generated by two independent Brownian motions in $\mathbb{R}^d$ intersect or, more generally, are within a short distance a of each other. The well known behavior of that function of a-above, below, and at the critical dimension $d=4$, as well as further corrections, are derived here by means of a single renormalization group equation. The equation's derivation is expected to shed some light on the β-function of the $\lambda\phi_d^4$ quantum field theory.

1. Introduction

The Brownian motion in addition to its own attraction has been a source of insight in various branches of mathematical analysis and physics. In particular, its properties have been quoted in the analysis of various "random walk" and "random surface", expansions used in statistical mechanics and quantum field theory. Since it is often the case that the insight is based on an analogy – rather than a complete reduction, the interest is not confined to the results, but extends also to the methods by which these can be proven. Thus we expect that something may still be gained from new derivations of old results (some of which are improved here).

With these comments in mind, we present here a new method for the study of the probabilities of intersection, and almost-intersection, for the *paths* of two independent Brownian motions: above, below and *at* the critical dimension $d=4$. In contrast with the previously available proofs, the analysis is based on a single equation – whose derivation requires only the more basic properties of the Bronian motion. In particular, a stronger use is made of the scale covariance – by means of which a "β-function" is derived for this problem.

* Sloan Foundation Research Fellow. Research supported in part by NSF grant PHY-8301493