Commun. Math. Phys. 96, 387–407 (1984)

Communications in
Mathematical
Physics

Nahm's Equations and the Classification of Monopoles

S. K. Donaldson

All Souls College, Oxford, United Kingdom and, The Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract. Solutions of Nahm's system of ordinary differential equations are produced by variational methods. This leads to an explicit parametrisation of the solutions to the Bogomolny equation over $\mathbb{R}^3$.

Introduction

The Bogomolny equations are a non-linear system of partial differential equations for a connection and Higgs field defined on a bundle over $\mathbb{R}^3$. With suitable boundary conditions at infinity the solutions, mathematical *monopoles*, fall into discrete classes indexed by a "topological charge." If, as we shall always suppose, the group of the bundle is SU(2) this charge is an integer $k \geqq 0$. Within each topological type the solutions are parametrised by continuous variables or moduli spaces M_k.

These monopoles have been studied from a number of different points of view. In the simplest case, taking $k = 1$, there is an explicit "fundamental solution" due to Prasad and Sommerfield which exhibits a qualitative soliton or particle-like structure. More generally Taubes has shown ([8] Chap. 4) that, for each value of the charge k, solutions to the equations exist which are approximate superpositions of k copies of the Prasad–Sommerfield monopole centred on widely separated points in $\mathbb{R}^3$.

On the other hand Hitchin and Nahm have developed independent but closely related methods of constructing monopoles stemming from twistor geometry [5]. In the final formulation of Hitchin the construction of a monopole becomes equivalent to finding an algebraic curve with certain properties, and this method was used by Hurtubise [7] to describe all monopoles of charge 2.

However neither from the analytic nor the geometric points of view was it possible to immediately identify the full moduli space M_k for larger values of k. This is the problem that we take up here, and we shall verify a conjecture of Atiyah and Murray [11] relating these moduli spaces to the rational functions on the Riemann sphere, $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.

The manifolds M_k, for $k \geqq 1$, have dimension $4k - 1$. It turns out to be more convenient to describe a slightly larger space $\tilde{M}_k$, of dimension $4k$, which is a circle bundle over M_k. The description in terms of rational functions is then: