Commun. Math. Phys. 95, 257–276 (1984)

Communications in
Mathematical
Physics

Eigenvalue Inequalities for Fermions in Gauge Theories

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Abstract. We show that QCD with a sufficient number of fermions of zero bare mass has physical massless particles. That result also follows from triangle anomalies, so only our method is novel. Our method involves proving special cases of recently conjectured paramagnetic inequalities for fermions. The proofs are simple applications of the Atiyah-Patodi-Singer theorem on spectral flow.

I. Introduction

Despite intensive study over the years, many fundamental properties of quantum chromodynamics – and strongly interacting gauge theories in general – are imperfectly understood. These include questions of chiral symmetry breaking, dynamical mass generation, and confinement.

There has recently been progress in understanding the properties of strongly interacting gauge theories through the development of rigorous inequalities [1–6]. In particular [2, 3], a surprising amount of information about symmetry realization in parity-conserving, vector-like gauge theories (like QCD) follows from relatively simple facts about fermion determinants and propagators in Euclidean space.

In these theories, after integrating out the fermion fields, we are left with a measure for the gauge field integration

$$d\mu(A)=\frac{1}{Z}e^{-[(1/4g^2)\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu}]}\det(\not{D}+M)\pi dA_\mu^a(x)\,, \tag{1}$$

which is an ordinary, real, positive measure. (The positivity of the measure depends on the fact that the fermion determinant is positive in four dimensions for fermions in a real representation of the gauge group; see, for instance [3].) Any inequality

* Supported in part by the National Science Foundation Grant No. PHY80-19754
† Supported in part by DOE Grant No. DE-AC02-76ER02220