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Integrable Graded Manifolds and Nonlinear Equations

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Abstract. A method is proposed for the classification of integrable embeddings of (2+2)-dimensional supermanifolds $V_{2|2}$ into an enveloping superspace supplied with the structure of a Lie superalgebra. The approach is first applied to the "even part" of the scheme, i.e. for the embeddings of 2-dimensional manifolds V_2 into Riemannian or non-Riemannian enveloping space. The general consideration is also illustrated by the example of superspaces supplied with the structure of the series $\mathrm{sl}(n, n+1)$, whose integrable supermanifolds are described by supersymmetrical 2-dimensional Toda lattice type equations. In particular, for $n=1$ they are described by the supersymmetrical Liouville and Sine-Gordon equations.

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This paper is mainly devoted to a construction for classifying integrable embeddings of (2+2)-dimensional supermanifolds $V_{2|2}$ into the enveloping superspace $V_{N|M}$ supplied with the structure of a finite-dimensional Lie superalgebra $\mathfrak{G}=\mathfrak{G}_{\bar{0}}\oplus\mathfrak{G}_{\bar{1}}$ (with the product [,]), whose $\mathbb{Z}$-grading $\left(\mathfrak{G}=\bigoplus_{m\in\mathbb{Z}}\mathfrak{G}_m, [\mathfrak{G}_m, \mathfrak{G}_n]\subset\mathfrak{G}_{m+n}\right)$ is consistent with the $\mathbb{Z}_2$-grading, i.e. $\mathfrak{G}_{\bar{0}}=\oplus\mathfrak{G}_{2m}$, $\mathfrak{G}_{\bar{1}}=\oplus\mathfrak{G}_{2m+1}$.

Henceforth we use the following definitions. Denote

$$\mathfrak{R}_{\mathfrak{A}\mathfrak{B}}(Z;\mathfrak{J})\equiv[\partial/\partial Z^{\mathfrak{A}}+\mathfrak{J}_{\mathfrak{A}}(Z), \partial/\partial Z^{\mathfrak{B}}+\mathfrak{J}_{\mathfrak{B}}(Z)], \tag{1.1}$$

$1\leqq\mathfrak{A}<\mathfrak{B}\leqq\mathfrak{N}+\mathfrak{M}$, where $\mathfrak{J}_{\mathfrak{A}}\equiv\sum_{1\leqq\kappa\leqq\dim\mathfrak{G}} a^{\kappa}_{\mathfrak{A}}(Z)F_{\kappa}$ are some functions of $Z^{\mathfrak{A}}$ taking values in the Grassmann hull $\mathfrak{G}(\Lambda)$ of the superalgebra $\mathfrak{G}$ with the basis $F_{\kappa}, \mathfrak{J}_{\mathfrak{A}}\dot{\in}\mathfrak{G}(\Lambda)$; $Z^A=y^A$, $1\leqq A\leqq\mathfrak{N}$, are usual Cartesian coordinates, $Z^{\Omega+\mathfrak{N}}=\Theta^{\Omega}$, $1\leqq\Omega\leqq\mathfrak{M}$, are canonical generators of the Grassmann algebra $\Lambda_{\mathfrak{M}}$.

By a grading spectrum of $A_{\pm}$ we understand the choice of a $\mathbb{Z}$-grading of the superalgebra $\mathfrak{G}$ and the condition that for $\mathfrak{N}=2$, $\mathfrak{M}=0$ the operators $A_{\pm}(z_+, z_-)$