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Existence, Uniqueness, and Nondegeneracy of Positive Solutions of Semilinear Elliptic Equations

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Abstract. We study positive solutions of the Dirichlet problem: $\Delta u(x)+f(u(x))=0$, $x\in D^n$, $u(x)=0$, $x\in\partial D^n$, where D^n is an n-ball. We find necessary and sufficient conditions for solutions to be nondegenerate. We also give some new existence and uniqueness theorems.

In this paper we study positive solutions of the Dirichlet problem

$$\Delta u(x)+f(u(x))=0\,,\qquad x\in\Omega\,, \tag{1}$$

$$u(x)=0\,,\qquad x\in\partial\Omega\,, \tag{2}$$

where Ω is an n-ball D^n_R of radius R. Our original interest was with the degeneracy problem for solutions of (1), (2). That is, we wanted to find conditions under which 0 is not in the spectrum of the linearized equations; in symbols,

$$\text{if}\ \begin{Bmatrix}\Delta v(x)+f'(u(x))v(x)=0\,, & x\in\Omega\\ v(x)=0\,, & x\in\partial\Omega\end{Bmatrix},\quad\text{then}\quad v\equiv 0\,.$$

When this holds, we say that the solution u of (1), (2) is *non-degenerate*; otherwise u is called *degenerate*. The interest in this notion comes from the fact that the non-degeneracy of a solution allows application of certain topological techniques to it, whereby its stability properties can be investigated [8, Chap. 24, Sect. D]. In pursuing this problem, we were led quite naturally to existence and uniqueness questions for (1), (2), and we also obtain some new results in these directions.

From a result of Gidas et al. [4], all positive solutions of (1), (2) on $\Omega=D^n_R$ are (monotone decreasing) functions of the radius, and must therefore satisfy a non-autonomous ordinary differential equation. Our uniqueness results follow from a general theorem concerning non-bifurcation of solutions of equations of the form

$$u''+g(u,u',t)=0\,, \tag{3}$$

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