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Scaling Limit of the Energy Variable for the Two-Dimensional Ising Ferromagnet

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Abstract. The critical point limit law (scaling limit) of the suitably renormalized energy variable is explicitly calculated for the two-dimensional nearest-neighbour Ising cylinder with free edges. It is shown that the renormalization factor has to behave as $(2M\,2N \ln N)^{1/2}$, where $2M$ denotes the number of rows and $2N$ the number of columns. By first taking the limit $M \to \infty$ and then $N \to \infty$, the limit law is proven to be Gaussian.

1. Introduction

Let us consider d-dimensional Ising ferromagnets with pairwise interactions. The Gibbs measure in a finite volume $\Lambda \subset \mathbb{Z}^d$ for a given configuration $\{\sigma_\Lambda\}$ at temperature β^{-1} and external field h will be taken as

$$d\mu_\Lambda(\beta, h; \{\sigma_\Lambda\}) = Z_\Lambda^{-1}(\beta, \beta h) \exp\left\{\beta \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j + \beta h \sum_{i\in\Lambda} \sigma_i\right\} \prod_{k\in\Lambda} d\rho(\sigma_k), \tag{1}$$

$$d\rho(\sigma) = \tfrac{1}{2}[\delta(\sigma - 1) + \delta(\sigma + 1)], \tag{2}$$

where $J_{ij} \geqq 0$ is such that the thermodynamic limit exists and $Z_\Lambda(\beta, \beta_h)$ is the partition function normalizing the Gibbs measure. Equation (1) defines the joint probability distribution of the $|\Lambda|$ spins in the block Λ as usual for d-dimensional Ising models.

Let us define the random variables magnetization M_Λ and energy E_Λ for a spin block Λ by

$$M_\Lambda = \sum_{i\in\Lambda} S_i, \tag{3}$$

$$E_\Lambda = -\sum_{i,j\in\Lambda} J_{ij} S_i S_j, \tag{4}$$

where S_i denotes the random variable associated to the i^{th} spin.

Up to this point, every concept has been introduced for *one* block spin $\Lambda \subset \mathbb{Z}^d$. One may then ask the following question: what happens if one considers block