Commun. Math. Phys. 94, 293–315 (1984)

Communications in Mathematical Physics

Conservative Diffusions

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Abstract. In Nelson's stochastic mechanics, quantum phenomena are described in terms of diffusions instead of wave functions. These diffusions are formally given by stochastic differential equations with extremely singular coefficients. Using PDE methods, we prove the existence of solutions. This result provides a rigorous basis for stochastic mechanics.

Introduction

In Nelson's stochastic mechanics quantum phenomena are described in terms of diffusions instead of wavefunctions. The class of diffusions considered here arises in stochastic mechanics; and in order to motivate our theorems, we provide a brief description of the theory. There is unfortunately a certain amount of technical material involved, so our description here must be illustrative rather than precise. In the main body of the paper we will be precise.

Consider a point particle in $\mathbb{R}^3$ moving under the influence of a potential $V(x)$. When we specify what we mean by moving—the kinematics, and what we mean by influence of a potential—the dynamics, we will have a mechanics.

The kinematical proposal is that we describe the motion of our particle with a Markovian diffusion $\xi(t)$ in $\mathbb{R}^n$. (We define all terms from diffusion theory we use at the start of Sect. I.) To get agreement with quantum mechanics, we must specify the size of the fluctuations; we require that on any interval $[a, b]$, the quadratic variation of our process is $(\hbar/2m)(b-a)$ a.s., where $\hbar$ is Plank's constant over 2π and m is the mass of the particle. Notice this has the correct units for a quadratic variation, and henceforth set $\hbar = m = 1$. We will not discuss the possible physical natures of these fluctuations—see Nelson [1] for a conjecture—but will simply remark that if we assume them to be manifestations of an isotropic translation invariant phenomenon, then the second order part of the generator of our diffusion must be $\frac{1}{2}\Delta$. So, the diffusions to be considered are those with a time dependent generator of the form $\frac{1}{2}\Delta + b(x, t)\cdot\nabla$.

The dynamics is given by the Guerra–Morato–Nelson variational principle. Fix a finite time interval $[0, T]$, an initial density $\rho_0(x)$ and an initial forward drift $b_0(x)$. Consider the class of diffusions with generator of the form $\frac{1}{2}\Delta + b(x, t)\cdot\nabla$, where $b(x, 0) = b_0(x)$ and with $\xi(0)$ having density $\rho_0(x)$. For such a diffusion $\xi(t)$,