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New Integrable Problem of Classical Mechanics

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Abstract. Complete integrability in Liouville's sense is proven for rotation of an arbitrary rigid body with a fixed point in a Newtonian field with an arbitrary homogeneous quadratic potential. A consequence is the complete integrability of rotation of a rigid body with fixed center of mass in the field of arbitrary sufficiently remote objects (in the second approximation). Explicit formulae are obtained expressing angular velocities of the rigid body in terms of θ-functions for Riemannian surfaces. Integrable cases are found for rotation of a rigid body in nonlinear Newtonian potential fields.

1. Introduction and Summary

Investigation of dynamics of a rigid body with a fixed point in a Newtonian potential field $\varphi(x^1, x^2, x^3)$ is a classical problem of mechanics. For a linear potential, $\varphi = a_1 x^1 + a_2 x^2 + a_3 x^3$, rotation of a rigid body is described by the Euler-Poisson equations, that are not integrable in the general case. The dynamics is integrable only in three special cases which were discovered by Euler [1], Lagrange [2], and Kowalewski [3]. In the two former cases the problem is integrated in terms of elliptic functions, in the Kowalewski case – in terms of the Riemann θ-functions of two variables.

The main result of the present work is that rotation of an arbitrary rigid body with a fixed point in a Newtonian field given by an arbitrary quadratic potential, $\varphi = \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} x^i x^j$, is always completely integrable in Liouville's sense. The dynamical equations are integrated explicitly in terms of the Riemann θ-functions of 4 variables, restricted to a three-dimensional manifold (a Prym variety).

The problem of rotation of a rigid body in a quadratic potential field appears naturally in the following situation. Let us consider an arbitrary rigid body T fixed at its center of mass O under the action of the gravitational field of an object V (consisting, say, of several disconnected massive bodies). Suppose l is a linear size of the body T, R is the minimal distance from the point O to the field source V, and