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Uniqueness and a Priori Bounds for Certain Homoclinic Orbits of a Boussinesq System Modelling Solitary Water Waves

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Abstract. This paper establishes surprisingly precise a priori bounds on the L_∞-norm of certain singular solutions of a system of two nonlinear Sturm-Liouville equations which model solitary water waves.

These solutions can be interpreted as homoclinic orbits for a system of four first order ordinary differential equations. The uniqueness of these homoclinic orbits is established for certain choices of a parameter c, the phase speed of the waves. These observations do not result from perturbation of linear theory, but are global.

I. Introduction

The present paper sets out to further analyse the set of solitary wave solutions of the equations of Boussinesq type which Bona and Smith [1] introduced to model long water waves in a channel. In earlier papers [3–5] it was shown that these equations,

$$\eta_t + u_x + (u\eta)_x - \tfrac{1}{3}\eta_{xxt} = 0\,,$$

$$u_t + \eta_x + uu_x - \tfrac{1}{3}(u_t + \eta_x)_{xx} = 0\,,$$

have travelling solitary wave solutions $(u(x-ct), \eta(x-ct))$ for each value of the wave speed c with $c > 1$. These solutions satisfy the boundary-value problem

$$\left.\begin{array}{ll} c(u - \tfrac{1}{3}u'') = \eta - \tfrac{1}{3}\eta'' + \tfrac{1}{2}u^2 & \text{on } \mathbb{R}\,, \\ c(\eta - \tfrac{1}{3}\eta'') = u + u\eta & \text{on } \mathbb{R}\,, \\ u' < 0\,, \quad \eta' < 0 & \text{on } (0, \infty)\,, \\ \lim\limits_{x\to\infty} u(x) = \lim\limits_{x\to\infty} \eta(x) = 0\,, & u, \eta \text{ even on } \mathbb{R}\,. \end{array}\right\} \qquad (*)$$

For fixed c they correspond to certain homoclinic orbits joining the rest point $(0, 0)$ to itself.