Commun. Math. Phys. 94, 141–153 (1984)

Communications in Mathematical Physics

Amount of Rotation About a Point and the Morse Index*

John N. Mather

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Abstract. For the case of exact, area preserving, monotone twist diffeomorphisms, we give formulas relating the amount of rotation about an orbit and certain Morse indices.

1. Introduction

This paper continues the study of exact, area preserving, monotone twist diffeomorphisms of the annulus which I have pursued in [14–19]. I should point out that Aubry earlier found ideas similar to many of those in [14–19] (see [1]). See [3] and the references therein for later developments in Aubry's theory. See also Katok [11, 12] for other recent developments.

Precise definitions of the terminology used in this introduction will be given in Sects. 2–4.

An exact, area preserving, monotone twist diffeomorphism f (Sect. 2) admits a definition in terms of a global generating function. It is then possible to give a variational formulation (Sect. 3) of periodic orbits: periodic orbits of type (p, q) are in one-one correspondence with critical points of an "energy" functional W^q defined on the space $\mathscr{X}_{pq}$ of "states" of type (p, q). Our first result relates the amount of rotation ϱ about a periodic orbit of type (p, q) and the Morse index I of W^q at the corresponding critical point x in $\mathscr{X}_{pq}$.

In order to state this result, we now give the definition of ϱ for the case of a C^1 diffeomorphism f of $T \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, which is isotopic to the identity and a fixed point P of f.

Definition of the Amount of Rotation of f About a Fixed Point P. Let τ_P denote the tangent space to $T \times \mathbb{R}$ at P. Each ray emanating from the origin in τ_P intersects the unit circle in τ_P in exactly one point. Hence, we may identify the set R_P of such rays with the unit circle, and provide R_P with the topology which makes this identification a homeomorphism. The derivative $df_P : \tau_P \to \tau_P$ induces a

* Partially supported by NSF contract No. MSC-82-01604