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Remarks on the Breakdown of Smooth Solutions for the 3-*D* Euler Equations

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Abstract. The authors prove that the maximum norm of the vorticity controls the breakdown of smooth solutions of the 3-*D* Euler equations. In other words, if a solution of the Euler equations is initally smooth and loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches; equivalently, if the vorticity remains bounded, a smooth solution persists.

The motion of an ideal incompressible fluid is governed by a system of partial differential equations known as the Euler equations. For two-dimensional flow, solutions of the Euler equations with smooth initial data remain smooth for all time. However, in three space dimensions several numerical investigations ([2, 3, 11]) predict very different phenomena. In particular, these computations suggest that solutions of the fluid equations which at first represent smooth flows may develop singularities, and furthermore that this breakdown of regularity signifies the onset of turbulent behavior. Qualitative arguments and numerical experiments indicate that the formation of singularities is related to the concentration of vorticity on successively smaller sets ([3, 4]). In this note we establish a mathematically rigorous link between the accumulation of vorticity and the formation of singularities for the 3-*D* Euler equations: we show that, if a solution is initially smooth and loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches. Therefore, it is not possible for other kinds of singularities (such as those in the deformation tensor or even milder singularities) to form before the vorticity becomes unbounded. In other words, the maximum norm of the vorticity alone controls the breakdown of smooth solutions for the 3-*D* Euler equations.

Euler's equations for the motion of an incompressible, inviscid fluid in free

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