Commun. Math. Phys. 93, 453–460 (1984)

Communications in
Mathematical
Physics

Instantons and Geometric Invariant Theory

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Abstract. We show that the Yang–Mills instantons can be described in terms of certain holomorphic bundles on the projective plane. The proof uses explicit matrix descriptions arising from monads and an analysis of the corresponding groups of symmetries.

In this note we show that the parameter spaces of instantons on $\mathbb{R}^4$ can be given a purely *complex* algebraic description. Precisely, let G be one of the classical groups $SU(\ell)$, $SO(\ell)$, $Sp(\ell)$, and $k \geqq 0$ be minus the Pontrayagin index (we work with anti self-dual connections) of a G bundle P over $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Denote by $\tilde{M}(G,k)$ the moduli space whose points represent isomorphism classes of pairs:

$$(\text{anti self-dual } G\text{-connection on } P, \text{ isomorphism } P_\infty \cong G).$$

These "framed" moduli spaces are in some respects more natural than the usual moduli of connections alone: $M(G,k) = \tilde{M}(G,k)/G$, and are manifolds of dimension $4\ell k$, $4(\ell-2)k$, $4(\ell+1)k$ respectively, for k sufficiently large.

Now fix a complex structure on $\mathbb{R}^4$ compatible with the metric and compactify in a different way to the complex projective plane:

$$\mathbb{R}^4 \cong \mathbb{C}^2 \subseteq \mathbb{CP}^2 = \mathbb{C}^2 \cup \ell_\infty .$$

Consider analogously the moduli space:

$$\mathcal{O}\tilde{M}(G^{\mathbb{C}}, k)$$

of holomorphic bundles on $\mathbb{CP}^2$ for the associated complex group, trivial on the line at ∞, ℓ_∞ and with a fixed holomorphic trivialization there. Then we shall prove:

Theorem. *There is a natural one-to-one correspondence*:

$$\tilde{M}(G,k) \underset{R}{\overset{\cong}{\rightarrow}} \mathcal{O}\tilde{M}(G^{\mathbb{C}},k).$$

(One consequence of this correspondence is that, interchanging the evident symmetry groups, we get an action of $SO(4)$ on $\mathcal{O}\tilde{M}$ and of $SL_2(\mathbb{C})$ on $\tilde{M}$.)