Commun. Math. Phys. 93, 417–436 (1984)

Communications in Mathematical Physics
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Integrable Euler Equations on SO(4) and their Physical Applications

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Abstract. For the Lie algebra SO(4) (and other six dimensional Lie algebras) we find some Euler's equations which have an additional fourth order integral and are algebraically integrable (in terms of elliptic functions) in a one parameter set of orbits. Integrable Euler's equations having an additional second order integral and generalizing Steklov's case are presented. Equations for rotation of a rigid body having n ellipsoid cavities filled with the ideal incompressible fluid being in a state of homogeneous vortex motion are derived. It is shown that the obtained equations are Euler's equations for the Lie algebra of the group $G_{n+1}=\mathrm{SO}(3)\times\ldots\times\mathrm{SO}(3)$. New physical applications of Euler's equations on SO(4) are discussed.

1. Introduction and Summary

We consider two classes of six-dimensional Lie algebras L, which are specified by the following commutation relations, that are written down in terms of a basis X_i, Y_k $(i,j,k=1,2,3)$, in the first class, A,

$$[X_i,X_j]=\varepsilon_{ijk}n_kX_k\,,\qquad [X_i,Y_j]=\varepsilon_{ijk}n_kY_k\,,$$
$$[Y_i,Y_j]=\varepsilon_{ijk}n_k\kappa X_k\,,\tag{1.1}$$

and in the second class, B

$$[X_i,X_j]=\varepsilon_{ijk}n_kX_k\,,\qquad [X_i,Y_j]=0\,,\qquad [Y_i,Y_j]=\varepsilon_{ijk}m_kY_k\,.\tag{1.2}$$

Here ε_{ijk} is the totally skew-symmetric tensor, and n_k, m_k, κ are structure constants. The following Lie algebras belong to class A: SO(4) $(n_i=1,\ \kappa=1)$, SO(3, 1) $(n_1=n_2=1,\ n_3=-1,\ \kappa=-1)$, SO(2, 2) $(n_1=n_2=1,\ n_3=-1,\ \kappa=1)$, E_3 $(n_i=1,\ \kappa=0)$, L_3 $(n_1=n_2=1,\ n_3=-1,\ \kappa=0)$ etc. The Lie algebras E_3 and L_3 are those corresponding to the groups of motion of the three-dimensional Euclidean and pseudo-Euclidean spaces, respectively. The Lie algebras belonging to class B are $\mathrm{SO}(4)=\mathrm{SO}(3)+\mathrm{SO}(3)$ $(n_i=1,\ m_i=1)$, $\mathrm{SL}(2,R)+\mathrm{SL}(2,R)$ $(n_1=m_1=n_2=m_2=1,\ n_3=m_3=-1)$, $\mathrm{SO}(3)+\mathrm{SL}(2,R)$ $(n_i=1,\ m_1=m_2=1,\ m_3=-1)$ etc.