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Quantum Ito's Formula and Stochastic Evolutions*

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Abstract. Using only the Boson canonical commutation relations and the Riemann-Lebesgue integral we construct a simple theory of stochastic integrals and differentials with respect to the basic field operator processes. This leads to a noncommutative Ito product formula, a realisation of the classical Poisson process in Fock space which gives a noncommutative central limit theorem, the construction of solutions of certain noncommutative stochastic differential equations, and finally to the integration of certain irreversible equations of motion governed by semigroups of completely positive maps. The classical Ito product formula for stochastic differentials with respect to Brownian motion and the Poisson process is a special case.

1. Introduction

We construct a quantum mechanical generalisation of the Ito-Doob theory of mean-square stochastic integration and an associated Ito product formula in which Brownian motion is replaced by the pair of operator processes $(A_f(t):t\geqq 0)$, $(A_g^\dagger(t):t\geqq 0)$, where $A_f(t)=a(f\chi_{[0,t]})$, and $A_g^\dagger(t)=a^\dagger(g\chi_{[0,t]})$ are annihilation and creation operators in the Boson Fock space $\Gamma(\mathfrak{h})$ over $\mathfrak{h}=L^2[0,\infty)\otimes\mathfrak{k}$, $\mathfrak{k}$ being a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$, and the Poisson process is replaced by what we call a gauge process $(\Lambda_\Pi(t):t\geqq 0)$, where π is a locally bounded self adjoint operator valued map from $[0,\infty)$ to $B(\mathfrak{k})$ and $\Lambda_\pi(t)$ is the differential second quantisation of $I\otimes\Pi(t)$. This leads to a stochastic calculus which is in some respects simpler and more natural than the classical theory, which is contained as a special case.

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