Algebraic Properties of Cellular Automata

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Abstract. Cellular automata are discrete dynamical systems, of simple construction but complex and varied behaviour. Algebraic techniques are used to give an extensive analysis of the global properties of a class of finite cellular automata. The complete structure of state transition diagrams is derived in terms of algebraic and number theoretical quantities. The systems are usually irreversible, and are found to evolve through transients to attractors consisting of cycles sometimes containing a large number of configurations.

1. Introduction

In the simplest case, a cellular automaton consists of a line of sites with each site carrying a value 0 or 1. The site values evolve synchronously in discrete time steps according to the values of their nearest neighbours. For example, the rule for evolution could take the value of a site at a particular time step to be the sum modulo two of the values of its two nearest neighbours on the previous time step. Figure 1 shows the pattern of nonzero sites generated by evolution with this rule from an initial state containing a single nonzero site. The pattern is found to be self-similar, and is characterized by a fractal dimension $\log_2 3$. Even with an initial state consisting of a random sequence of 0 and 1 sites (say each with probability $\frac{1}{2}$), the evolution of such a cellular automaton leads to correlations between separated sites and the appearance of structure. This behaviour contradicts the second law of thermodynamics for systems with reversible dynamics, and is made possible by the irreversible nature of the cellular automaton evolution. Starting from a maximum entropy ensemble in which all possible configurations appear with equal probability, the evolution increases the probabilities of some configurations at the expense of others. The configurations into which this concentration occurs then dominate ensemble averages and the system is "organized" into having the

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