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On Characteristic Exponents in Turbulence

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Abstract. Ruelle has found upper bounds to the magnitude and to the number of non-negative characteristic exponents for the Navier-Stokes flow of an incompressible fluid in a domain Ω. The latter is particularly important because it yields an upper bound to the Hausdorff dimension of attracting sets. However, Ruelle's bound on the number has three deficiences: (i) it relies on some unproved conjectures about certain constants; (ii) it is valid only in dimensions $\geqq 3$ and not 2; (iii) it is valid only in the limit $\Omega \to \infty$. In this paper these deficiences are remedied and, in addition, the final constants in the inequality are improved.

Ruelle [1] has derived upper bounds on the magnitude and number of non-negative characteristic exponents of the Navier-Stokes equation for the flow of an incompressible fluid in a domain $\Omega \in \mathbb{R}^d$. The bound on the number, $\tilde{N}(\mu)$ [defined in (42)], is particularly interesting because it leads to an upper bound on the Hausdorff dimension of a compact attracting set [1, Corollary 2.3]. Unfortunately, the bounds in [1] on $\tilde{N}(\mu)$, unlike those on the magnitude, have certain deficiencies which are

(i) They rely for their validity on some conjectured, but as yet unproved, relations between the sharp constants in two known inequalities.

(ii) They are valid only for $d \geqq 3$.

(iii) Because Weyl's asymptotic formula for the eigenvalues of the Laplacian in Ω is used, the inequalities are not valid for any fixed Ω, but only in the limit $\Omega \to \infty$.

In this paper a different proof of Ruelle's inequality for the number will be given so that the above three deficiencies are remedied. The result is contained in Eqs. (40)–(43).

Let $v: \Omega \to \mathbb{R}^d$ denote a solution to the Navier-Stokes equation, and let $\mu_1 \geqq \mu_2 \geqq \ldots$ be the characteristic exponents corresponding to a probability measure $\varrho(dv)$ on the space of solutions that is ergodic with respect to the Navier-

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