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On the Curvature of Piecewise Flat Spaces

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Abstract. We consider analogs of the Lipschitz-Killing curvatures of smooth Riemannian manifolds for piecewise flat spaces. In the special case of scalar curvature, the definition is due to T. Regge; considerations in this spirit date back to J. Steiner. We show that if a piecewise flat space approximates a smooth space in a suitable sense, then the corresponding curvatures are close in the sense of measures.

0. Introduction

Let X^n be a complete metric space and $\Sigma \subset X^n$ a closed, subset of dimension less than or equal to $(n-1)$. Assume that $X^n \backslash \Sigma$ is isometric to a smooth (incomplete) n-dimensional Riemannian manifold. How should one define the curvature of X^n at points $x \in \Sigma$, near which the metric need not be smooth and X need not be locally homeomorphic to $U \subset R^n$? In [C, Wi], this question is answered (in seemingly different, but in fact, equivalent ways) for the Lipschitz-Killing curvatures, and their associated boundary curvatures under the assumption that the metric on X^n is piecewise flat. The precise definitions (which are given in Sect. 2), are formulated in terms of certain "angle defects." For the mean curvature and the scalar curvature they are originally due to Steiner [S] and Regge [R], respectively.

It is worth noting at the outset that the discussion of curvature at nonsmooth points depends in a crucial way on the precise notion of curvature under consideration. If, for example, one wishes to generalize the Pontrjagin forms, the notion of "angle defect" will no longer suffice. It can be replaced by the much less elementary "η-invariant" (see [C]).

Recall that in the smooth case, the j^{th} Lipschitz-Killing curvature R^j is the measure on M^n, which is zero for j odd and which for j even is given by integrating

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