Commun. Math. Phys. 92, 203–215 (1983)

Communications in
Mathematical
Physics

Large Solutions for Harmonic Maps in Two Dimensions*

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Abstract. We seek critical points of the functional $E(u)=\int\limits_{\Omega}|\nabla u|^2$, where Ω is the unit disk in $\mathbb{R}^2$ and $u:\Omega\to S^2$ satisfies the boundary condition $u=\gamma$ on $\partial\Omega$. We prove that if γ is not a constant, then E has a local minimum which is different from the absolute minimum. We discuss in more details the case where $\gamma(x,y)=(Rx,Ry,\sqrt{1-R^2})$ and $R<1$.

Introduction

Let $\Omega=\{(x,y)\in\mathbb{R}^2;\ x^2+y^2<1\}$ and $S^2=\{(x,y,z)\in\mathbb{R}^2;\ x^2+y^2+z^2=1\}$. Let $\gamma:\partial\Omega\to S^2$ be given and assume that γ is the restriction to $\partial\Omega$ of some function in $H^1(\Omega;S^2)$[1]. We set

$$E(u)=\int\limits_{\Omega}|\nabla u|^2 \quad \text{for} \quad u\in H^1(\Omega;\mathbb{R}^3)$$

and

$$\mathscr{E}=\{u\in H^1(\Omega;S^2); u=\gamma \quad \text{on} \quad \partial\Omega\}.$$

We seek critical points of E on $\mathscr{E}$. It is obvious that there exists some $\underline{u}\in\mathscr{E}$ such that

$$E(\underline{u})=\operatorname*{Inf}_{\mathscr{E}} E.$$

Our first result is the following:

Theorem 1. *If γ is not a constant, there exists a critical point of E on $\mathscr{E}$ which is different from $\underline{u}$.*

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* Work partially supported by US National Science Foundation grant PHY-8116101-A01

1 We use the standard notation for Sobolev spaces:
$H^1(\Omega;\mathbb{R}^3)=\{u\in L^2(\Omega;\mathbb{R}^3);\quad u_x,u_y\in L^2(\Omega;\mathbb{R}^3)\}$ and
$H^1(\Omega;S^2)=\{u\in H^1(\Omega;\mathbb{R}^3);\quad u(x,y)\in S^2$ a.e. on $\Omega\}$