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On a C^*-Algebra Approach to Phase Transition in the Two-Dimensional Ising Model

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Abstract. We investigate the state on the C^*-algebra of Pauli spins on a one-dimensional lattice (infinitely extended in both directions) which gives rise to the thermodynamic limit of the Gibbs ensemble in the two-dimensional Ising model (with nearest neighbour interaction). It is shown that the representation of the Pauli spin algebra associated with the state is factorial above and at the known critical temperature, while it has a two-dimensional center below the critical temperature. As a technical tool, we derive a general criterion for a state of the Pauli spin algebra corresponding to a Fock state of the Fermion algebra to be primary. We also show that restrictions of two quasifree states of the Fermion algebra to its even part are equivalent if and only if the projection operators E_1 and E_2 (on the direct sum of two copies of the basic Hilbert space) satisfy the following two conditions: (1) $E_1 - E_2$ is in the Hilbert–Schmidt class, (2) $E_1 \wedge (\mathbf{1} - E_2)$ has an even dimension, where the even-oddness of dim $E_1 \wedge (\mathbf{1} - E_2)$ is called $\mathbb{Z}_2$-index of E_1 and E_2 and is continuous in E_1 and E_2 relative to the norm topology.

1. Main Results

We consider the two-dimensional Ising model with the Hamiltonian (with the free boundary condition)

$$H^{LM}(\xi) = -\left(\sum_{i=-L}^{L-1} \sum_{j=-M}^{M} J_1 \xi_{ij} \xi_{i+1,j} + \sum_{i=-L}^{L} \sum_{j=-M}^{M-1} J_2 \xi_{ij} \xi_{i,j+1} \right), \tag{1.1}$$

where $\xi_{ij} = \pm 1$ is the (classical) spin at the lattice site $(i,j) \in \mathbb{Z}^2$, and J_1 and J_2 are positive constants. We are interested in the thermodynamic limit ($L,M \to \infty$) of the Gibbs ensemble average

$$\langle F \rangle_{LM} = Z_{LM}^{-1} \sum_{\xi} F(\xi) e^{-\beta H^{LM}(\xi)}, \tag{1.2}$$

$$Z_{LM} = \sum_{\xi} e^{-\beta H^{LM}(\xi)},$$

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