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Cauchy Problems for the Conformal Vacuum Field Equations in General Relativity

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Abstract. Cauchy problems for Einstein's conformal vacuum field equations are reduced to Cauchy problems for first order quasilinear symmetric hyperbolic systems. The "hyperboloidal initial value" problem, where Cauchy data are given on a spacelike hypersurface which intersects past null infinity at a spacelike two-surface, is discussed and translated into the conformally related picture. It is shown that for conformal hyperboloidal initial data of class H^s, $s \geqq 4$, there is a unique (up to questions of extensibility) development which is a solution of the conformal vacuum field equations of class H^s. It provides a solution of Einstein's vacuum field equations which has a smooth structure at past null infinity.

1. Introduction

In contrast with the field equations of other gauge theories Einstein's field equations are not conformally invariant: rescalings of the metric create Ricci curvature. However, important substructures of the field and the field equations, the conformal Weyl tensor and the vacuum Bianchi identity, written as an equation for the rescaled Weyl tensor, are conformally invariant. It is this very particular behaviour of the field equations under conformal rescalings which allows one to impose conditions on the global conformal structure of the field without restricting the freedom to prescribe asymptotic initial data for the field. Global conditions of this type are inherent in Penrose's concept of an "asymptotically empty and simple space-time" [1]. The essential idea is to stipulate for a given space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$ the existence of a surface $\mathscr{I}$ such that $M = \tilde{M} \cup \mathscr{I}$ forms a manifold with three-dimensional boundary $\mathscr{I}$ and on M the existence of a function Ω such that $\Omega > 0$ on $\tilde{M}$; $\Omega \equiv 0, d\Omega \neq 0$ on $\mathscr{I}$ and such that $g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu}$ extends to a smooth ("non-physical") Lorentz metric on M. Further global requirements imply that $\mathscr{I}$ consists of two components $\mathscr{I}^-$ respectively $\mathscr{I}^+$ ("past respectively future null infinity") each being diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$. The appropriateness of the fall-off conditions implicit in these assumptions was suggested in particular by preceding in-