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The General Exact Bijective Continuous Solution of Feigenbaum's Functional Equation

Patrick J. McCarthy

Department of Mathematics, Bedford College, University of London, Regent's Park, London NW1 4NS, England

Abstract. Solutions $f:\mathbb{R}\to\mathbb{R}$ of Feigenbaum's functional equation $f(f(x)) = \alpha^{-1}f(\alpha x)$, where $\alpha \neq 0$ is a fixed real number, account for many of the fascinating properties of the behaviour of successive iterates of (one parameter families of) nonlinear maps. In connection with the phenomenon of intermittency, interesting families of exact solutions have recently been found (for $\alpha > 0$). These solutions can all be derived from continuous bijective solutions which are topologically equivalent to translations. In this paper, the *general* exact continuous bijective solution is found for any $\alpha \neq 0$, positive or negative. In particular, it is shown that, for *any* $\alpha \neq 0$, there are solutions which are *in*equivalent to translations. And it is shown that bijective solutions equivalent to translations exist only when $0 < \alpha < 1$. These results considerably enlarge the stock of available exact solutions of Feigenbaum's equation.

1. Introduction

It is a remarkable fact that, for a wide class of real valued functions g of a real variable, the recursion relations $x_{n+1} = Kg(x_n)$ exhibit a rich qualitative [1] and quantitative [2] behaviour which is essentially independent of the recursion function g. Feigenbaum [3] and others [4–7] have provided an explanation of the scaling and universal properties of the transition to chaos via period doubling transformations in terms of a functional equation

$$f_0(f_0(x)) = \alpha^{-1} f_0(\alpha x) \tag{F}$$

for a real valued function f_0 of a real variable, where $\alpha < 0$ is a fixed real number. Solutions of (F) are evidently fixed points, in an appropriate function space, of the transformation T defined by

$$(Tf)(x) = \alpha f(f(\alpha^{-1}x)), \tag{T}$$

and the universal properties are [3–7] derived from the behaviour of T near a fixed point in certain eigendirections in function space.