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Stable Standing Waves of Nonlinear Klein-Gordon Equations

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Abstract. In this paper we give sufficient conditions for the stability of the standing waves of least energy for nonlinear Klein-Gordon equations.

0. Introduction

In this paper we give sufficient conditions for the stability of standing waves of the nonlinear Klein-Gordon equation:

$$u_{tt}-\Delta u+u+f(|u|)\arg u=0\,, \qquad x\in\mathbb{R}^n, \qquad n>2\,, \tag{0.1}$$

or equivalently the steady-state solutions of the modulated equation:

$$u_{tt}+2i\omega u_t-\Delta u+(1-\omega^2)u+f(|u|)\arg u=0\,. \tag{0.2ω}$$

We show the stability of the standing waves of lowest energy in the energy norm. They are stable with respect to the lowest energy solution set of

$$-\Delta u+(1-\omega^2)u+f(|u|)\arg u=0\,. \tag{0.3ω}$$

The existence of solutions of (0.3ω) has already been shown in [9] and [10]. In the generality presented in Sect. I this problem was solved by Berestycki and Lions in [10]. The condition for stability is very simple. If we define

$$d(\omega)=1/2\int|\nabla\varphi_\omega|^2dx+(1-\omega^2)/2\int|\varphi_\omega|^2dx+\int G(|\varphi_\omega|)dx\,,$$

where $G'=f$ and φ_ω is a least energy solution of (0.3ω), then:

Theorem. *If $d(\omega)$ is strictly convex in a neighborhood of ω_0, then φ_{ω_0} is stable.*

Equation (0.1) arises in particle physics. It models the field equation for spin-0 particles [4]. The existence of stable standing waves has, until now, eluded any rigorous proof. Anderson [1] showed by numerical computation that these equations can have stable standing waves. He studied the particular example