Commun. Math. Phys. 91, 65–73 (1983)

Communications in Mathematical Physics

The $\frac{1}{r}$ Expansion for the Critical Multiple Well Problem

Raphael Høegh-Krohn* and Mohammed Mebkhout**

Centre de Physique Théorique, C.N.R.S. Luminy, Case 907, F-13288 Marseille Cedex 9, France

Abstract. We consider the critical multiple well problem

$$H = -\Delta + \sum_{i=1}^{n} V(x - rx_i),$$

where $-\Delta + V(x)$ has a zero energy resonance. We prove that all eigenvalues and resonances of H tending to zero as $1/r^2$ are analytic in $1/r$. We give an explicit equation for the lowest nonvanishing coefficient in the $1/r$ expansion for any of these eigenvalues or resonances and observe that H has infinitely many resonances tending to zero. For $n=2$ and $n=3$, we compute the coefficients explicitly and for $n=2$, we also give the next coefficient in the $1/r$ expansion.

1. Introduction

In this paper, we study the critical multiple well problem, i.e. the asymptotic behavior of the eigenfunctions and resonances of

$$H_r = -\Delta + \sum_{i=1}^{n} V(x - rx_i) \tag{1.1}$$

in $L_2(R^3)$ as $r \to \infty$, where V is a potential of compact support such that $-\Delta + V$ has a zero energy resonance. We find that there are infinitely many resonances and finitely many eigenvalues, which tend to zero as $r \to \infty$. For these resonances and eigenvalues we prove that they are analytic in $1/r$ and we give the corresponding $1/r$ expansion. The eigenvalue tending to zero for $n=2$ was studied by Klaus and Simon in [1] where they proved that this eigenvalue behaved like $E_0(r) = -\sigma_0^2 r^{-2} + O(r^{-3})$, where σ_0 is the unique real solution of $\sigma = e^{-\sigma}$. We extend

* Université de Provence, Centre Saint Charles, and Centre de Physique Théorique, C.N.R.S. Marseille

** Université d'Aix-Marseille II, Faculté des Sciences de Luminy and Centre de Physique Théorique, Section 2, C.N.R.S. Marseille