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Almost Periodic Schrödinger Operators

III. The Absolutely Continuous Spectrum in One Dimension

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Abstract. We discuss the absolutely continuous spectrum of $H=-d^2/dx^2+V(x)$ with V almost periodic and its discrete analog $(hu)(n)=u(n+1)+u(n-1)+V(n)u(n)$. Especial attention is paid to the set, A, of energies where the Lyaponov exponent vanishes. This set is known to be the essential support of the a.c. part of the spectral measure. We prove for a.e. V in the hull and a.e. E in A, H and h have continuum eigenfunctions, u, with $|u|$ almost periodic. In the discrete case, we prove that $|A| \leqq 4$ with equality only if $V=$const. If k is the integrated density of states, we prove that *on* A, $2k dk/dE \geqq \pi^{-2}$ in the continuum case and that $2\pi \sin \pi k dk/dE \geqq 1$ in the discrete case. We also provide a new proof of the Pastur-Ishii theorem and that the multiplicity of the absolutely continuous spectrum is 2.

1. Introduction

This paper discusses the theory of one dimensional stochastic Schrödinger operators and Jacobi matrices, that is $H=-d^2/d^2x+V_\omega(x)$ on $L^2(-\infty,\infty)$ and $u \mapsto (hu)(n)=u(n+1)+u(n-1)+V_\omega(n)u(n)$ on $\ell^2(Z)$, where V_ω is a stationary ergodic process on R or Z. This set includes the highly random case and also the almost periodic (a.p.) case. As we will explain, our theorems are vacuous in the highly random case and are only of interest in cases close to the almost periodic case. A major role will be played by the integrated density of states, $k(E)$, and the Lyaponov exponent, $\gamma(E)$, defined, e.g. in [2] or in [8] [in the latter, the rotation number $\alpha(E)=\pi k(E)$ is discussed].

In this paper, our primary goal will be to study the absolutely continuous (a.c.) spectrum. Much of what we do should be viewed as a development of themes of Moser [12], Johnson and Moser [8], and most especially Kotani [10] (see Simon [18] for Kotani theory in the Jacobi matrix case). Virtually all the theorems we

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