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Universal Properties of Maps of the Circle with ε-Singularities

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Abstract. Following the work of Collet, Eckmann, and Lanford on the Feigenbaum conjecture, we study the structure of the renormalization transformation introduced in [12] upon maps of the circle with critical points of the form $x|x|^{\varepsilon}$.

1. Introduction

There appears to be a remarkable relationship between the seemingly universal features of certain bifurcations which destroy invariant tori of dissipative systems with particular incommensurate frequencies and the scaling properties of certain families of analytic mappings of the circle $T^1 = \mathbb{R}/\mathbb{Z}$. An explanation of this in terms of a renormalization transformation $\mathcal{T}$ is proposed in [12] and [4]. The evidence for this is numerical. Following the work of Collet et al. on the Feigenbaum conjecture [3] we study the action of $\mathcal{T}$ on a space of analytic functions of $x|x|^{\varepsilon}$, where $\varepsilon \geqq 0$ is small. The physically interesting case is $\varepsilon = 2$.

1.1. Motivation

A *cubic critical map* is any analytic homeomorphism of the circle which has a single critical point which is cubic. We will be interested here in the behaviour of (analytic) diffeomorphisms and critical maps related to the rotation number

$$\sigma = (\sqrt{5} - 1)/2 = 1/(1 + 1/(1 + 1/1 + \dots .$$

Similar results hold for any rotation number with a periodic (or eventually periodic) continued fraction, but to ease the exposition we stuck to the simplest case here. The rational approximants to σ are the numbers q_n/q_{n+1}, where

$$q_0/q_1 = 1/1, \quad q_1/q_2 = 1/(1+1), \quad q_2/q_3 = 1/(1 + 1/(1+1)), \dots ,$$

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