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Canonical Maps Between Semidirect Products with Applications to Elasticity and Superfluids

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Abstract. It is shown that two canonical maps arising in the Poisson bracket formulations of elasticity and superfluids are particular instances of general canonical maps between duals of semidirect product Lie algebras.

1. Introduction

In the last years many models in classical physics have been shown to possess Poisson structures. In many examples this Poisson structure is the canonical one on the dual of a Lie algebra, sometimes supplemented by a two-cocycle. It turns out that in almost all such cases, the Lie algebra in question is a semidirect product.

Quite often the same physical system allows descriptions in different sets of variables, thus obtaining two Poisson structures for the same model. These structures are not equivalent but connected. Such double descriptions commonly occur, e.g., in systems coupled to the magnetic field by introducing magnetic potentials; magnetohydrodynamics is such a system. The relation between the two descriptions, when linear, is produced by a Lie algebra homomorphism (like in the above-mentioned magnetic case of magnetohydrodynamics, see [5, 8]). In this case the dual map of the Lie algebra homomorphism is naturally canonical between the two Poisson structures. However, it was recently [5, 6] observed experimentally that in two physical models, elastodynamics and superhydrodynamics, the transformation between the two Poisson structures, even though non-linear, is still canonical. We hasten to emphasize that this is not the standard case in the theory of finite dimensional Lie algebras: given two general Lie algebras $\mathfrak{H}$ and $\mathfrak{G}$, there are no natural non-linear canonical maps from $\mathfrak{G}^*$ to $\mathfrak{H}^*$. The problem of interpretation of the above-mentioned non-linear canonical maps is the topic of this paper. Our explanation turns out to be very natural and simple (see, e.g., Theorem 3.5 below): canonical maps with range the dual of a Lie algebra are realized as momentum maps. This underlying philosophy is closely related to [8].

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