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Self-Similar Universal Homogeneous Statistical Solutions of the Navier–Stokes Equations

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Abstract. In this note we consider a family of statistical solutions of the Navier–Stokes equations (i.e. time dependent solutions of the Hopf equation) which seem to constitute the rigorous mathematical framework for the theory of homogeneous turbulence [1], [13]. The main feature of these solutions is that they are the transforms under suitable scalings of the *stationary* statistical solutions of a new system of equations (the Eq. (2) below).

0. Introduction

The theory of fully developed turbulence is nearly universally believed to be essentially that of the evolution of statistical distributions of flows governed by the Navier–Stokes equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \ \ \nabla \cdot u = 0. \tag{1}$$

Although significant progress has been made in the last 15 years in the rigorous mathematical approach to this theory [2, 4, 6, 9, 12, 21,...], no concrete family of homogeneous statistical solutions of the Navier–Stokes equations was found, nor is there as yet a consistent way of connecting these solutions with the Kolmogorov spectral estimates. In this paper we show that there exists a natural family of homogeneous statistical solutions of the Navier–Stokes equations enjoying some properties of self similarity and universality (Sect. 3). These solutions are obtained by suitable scalings of the *stationary* homogeneous statistical solutions of the equations

$$\frac{\partial u}{\partial t} - \tfrac{1}{2}u - \tfrac{1}{2}(x \cdot \nabla) u - \Delta u + (u \cdot \nabla) u + \nabla q = 0, \ \ \nabla \cdot u = 0, \tag{2}$$

(see Sect. 5 below). Note that the stationary form of Eqs. (2) differs only slightly from the (still not well understood) equations

$$\tfrac{1}{2}u + \tfrac{1}{2}(x \cdot \nabla) u - \Delta u + (u \cdot \nabla) u + \nabla q = 0, \ \ \nabla \cdot u = 0, \tag{3}$$