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Supersymmetry and the Atiyah-Singer Index Theorem*

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Abstract. Using a recently introduced index for supersymmetric theories, we present a simple derivation of the Atiyah-Singer index theorem for classical complexes and its G-index generalization using elementary properties of quantum mechanical supersymmetric systems.

I. Introduction

The mathematical structure of supersymmetric field theories has recently been shown to be closely related to global questions of differential and algebraic geometry [1–3].

Given a supersymmetric quantum field theory, one can define a quantity, $\mathrm{Tr}(-1)^F$ (where F is the fermion number) [1] which when properly regularized, counts the number of bosonic states minus the number of fermionic states in the Hilbert space of the theory. The properties of supersymmetry however, imply that this supersymmetric index depends only on the zero energy states due to the fact that all non-zero energy states appear in bose-fermi pairs. Furthermore, $\mathrm{Tr}(-1)^F$ is invariant under continuous deformations of the hamiltonian, and therefore is a topological index for the full quantum theory [1].

The fact that $\mathrm{Tr}(-1)^F$ is a topological index was used in [1] among other things to relate the possibility of supersymmetry breaking for supersymmetric σ-models to the vanishing of certain topological invariants of the manifold on which the σ-model is defined.

In this paper, we show that if we consider quantum mechanical supersymmetric systems (i.e., field theories in $0+1$ dimensions), and use $\mathrm{Tr}(-1)^F$ as a topological invariant, it is possible to calculate the index density for the Atiyah-

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