Commun. Math. Phys. 90, 125–159 (1983)

Communications in Mathematical Physics

On a Third-Order Phase Transition

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Abstract. The asymptotic behaviour of random variables of the general form

$$\ln \sum_{i=1}^{\kappa^N} \exp(N^{1/p}\beta\zeta_i)$$

with independent identically distributed random variables ζ_i is studied. This generalizes the random energy model of Derrida. In the limit $N\to\infty$, there occurs a particular kind of phase transition, which does not incorporate a bifurcation phenomenon or symmetry breaking. The hypergeometric character of the problem (see definitions of Sect. 4), its Φ-function, and its entropy function are discussed.

1. Introduction

The great majority of solvable mathematical models of mean field type, which show phase transitions, are closely related to bifurcation problems in the order parameters. For example, many magnetic spin models are of this type. The general features of these models may be roughly summarized in the following way: The calculation of the free energy in the thermodynamic limit is equivalent to a large deviation problem, which by Laplace's method goes over to a variational principle for the free energy. At an extremum, the first derivative with respect to the order parameters must necessarily vanish. In most interesting, exactly soluble examples this condition implies a bifurcation phenomenon.

In this paper, we study a phase transition phenomenon of a completely different type. In fact, we shall see below that we have to do with a kind of *iterated* large deviation problem. It is this iteration which provokes the phase transition. On the other hand, the models are simple enough so that no additional bifurcation phenomena appear. Since the mean free energy is once, but not twice, continuously differentiable at the critical point, we may speak of a third-order phase transition.

We now describe the models in more detail. Let us consider a probability measure ϱ on $\mathbb{R}$, which has an exponentially decreasing tail distribution at $+\infty$.