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Reconstruction of Singularities for Solutions of Schrödinger's Equation

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Abstract. We determine the behavior in time of singularities of solutions to some Schrödinger equations on R^n. We assume the Hamiltonians are of the form $H_0 + V$, where $H_0 = 1/2\Delta + 1/2 \sum_{k=1}^{n} \omega_k^2 x_k^2$, and where V is bounded and smooth with decaying derivatives. When all $\omega_k = 0$, the kernel $k(t,x,y)$ of $\exp(-itH)$ is smooth in x for every fixed (t,y). When all ω_1 are equal but non-zero, the initial singularity "reconstructs" at times $t = \frac{m\pi}{\omega_1}$ and positions $x = (-1)^m y$, just as if $V = 0$; k is otherwise regular. In the general case, the singular support is shown to be contained in the union of the hyperplanes $\{x | x_{js} = (-1)^l js_{y_{js}}\}$, when $\omega_j t/\pi = l_j$ for $j = j_1, \ldots, j_r$.

0. Introduction

Let $H = H_0 + V$ be a Schrödinger operator on $L^2(R^n)$, where H_0 is one of the model Hamiltonians:

(1) $-1/2\, \Delta$ Free Particle,
(2) $-1/2\, \Delta + 1/2\, |x|^2$ Isotropic Oscillator,
(3) $-1/2\, \Delta + 1/2 \sum_{k=1}^{n} \omega_k^2 x_k^2$ Anisotropic Oscillator,

and where the perturbing potential V is a 0-symbol on R^n, i.e. $|\partial_x^\alpha v| \leqq C_\alpha (1 + |x|)^{-|\alpha|}$. Then H generates a one parameter group of unitary operators $U(t) = \exp - itH$, whose Schwarz kernels we denote by $k_V(t,x,y)$ (called "propagators"). Our goal is to determine the wave front sets of these $k_V(t,x,y)$ when (t,y) are held fixed. This is the essential step in finding out how $U(t)$ propagates singularities—or, more correctly, how $U(t)$ smooths out and later reconstructs singularities.

The main problem is that although these distributions are oscillatory integral ones, i.e. of the form

$$k(t,x,y) = \int a(t,x,y,\theta) e^{iS(t,x,y,\theta)} d\theta,$$

they are not Lagrangian distributions (cf. 4, 7). Consequently, $WF(k(t,\cdot,y)) \nsubseteq$