Universal Scaling Behaviour for Iterated Maps in the Complex Plane

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Abstract. According to the theory of Schröder and Siegel, certain complex analytic maps possess a family of closed invariant curves in the complex plane. We have made a numerical study of these curves by iterating the map, and have found that the largest curve is a fractal. When the winding number of the map is the golden mean, the fractal curve has universal scaling properties, and the scaling parameter differs from those found for other types of maps. Also, for this winding number, there are universal scaling functions which describe the behaviour as $n \to \infty$ of the Q_n^{th} iterates of the map, where Q_n is the nth Fibonacci number.

I. Introduction

During the last few years, considerable progress has been made towards understanding the onset of turbulence or chaos of dynamical systems by studying the properties of one- and two-dimensional maps of real variables. Scaling and renormalization group ideas have elucidated the appearance of universal quantities associated with these maps which are directly relevant to physical systems. In particular, the period doubling transitions have now been found experimentally in various systems with exponents in good agreement with the predictions of the scaling theory [1]. Also progress has been made in applying scaling and renormalization group concepts to one-dimensional maps of the unit circle, which are believed to be generic for transitions which exhibit mode-locking on the route towards chaos [2], and also to the area-preserving twist map relevant to the study of critical Kolmogorov-Arnold-Moser (KAM) trajectories [3]. Other routes to turbulence, such as the scenario described by Ruelle and Takens [4], are understood only qualitatively.

In this paper, we study the scaling properties of a map $z \to f(z)$, where z is a complex variable and $f(z)$ is complex analytic, and usually a polynomial. Apart from the fact that such a map is a natural extension of the well-studied quadratic map on the unit interval [1], we were motivated by finding numerically that under

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