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Nonlinear Schrödinger Equations and Simple Lie Algebras

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Abstract. We associate a system of integrable, generalised nonlinear Schrödinger (NLS) equations with each Hermitian symmetric space. These NLS equations are considered as reductions of more general systems, this time associated with a reductive homogeneous space. The nonlinear terms are related to the curvature and torsion tensors of the appropriate geometrical space. The Hamiltonian structure is investigated using "r-matrix" techniques and shown to be "canonical" for all these equations. Throughout the reduction procedure this Hamiltonian structure does not degenerate. Each of the above systems of equations is gauge equivalent to a generalised ferromagnet. Reductions of the latter are discussed in terms of the corresponding NLS type equations.

1. Introduction

In the past few years it was discovered how to generalise the Toda lattice equations and their generalisations [1] to two dimensions. These $2-D$ generalised Toda lattices take the form of a multicomponent, Lorentz invariant, Lagrangian field theory. To each simple Lie algebra $\mathfrak{g}$, there corresponds one such field theory [2–5]. The number of field components is equal to the rank of $\mathfrak{g}$. When $\mathfrak{g}=\mathrm{sl}(2,\mathbb{C})$, the single component satisfies the sine-Gordon equation.

This is just one of the "group theoretic" generalisations of the sine-Gordon equation. Others include the chiral model and the nonlinear sigma model, each of which can be associated with a given semi-simple Lie group. Various reductions of these occur by restricting the field components to lie on some homogenous space.

In this paper we carry out a similar generalisation and classification of the nonlinear Schrödinger (NLS) equation. Furthermore, to each such generalised NLS equation we give the corresponding generalised ferromagnet.

As is well known, the system of equations

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