Commun. Math. Phys. 89, 77–102 (1983)

Communications in Mathematical Physics

Quantization and the Uniqueness of Invariant Structures

William Arveson*

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract. We determine and classify certain algebraic structures, defined on the space of all complex-valued polynomials in $2n$ real variables, which admit *affine* contact transformations as automorphisms. These are the structures which have the minimum symmetry necessary to define the basic linear and angular momentum observables of classical and quantum mechanics. The results relate to the so-called Dirac problem of finding an appropriate mathematical characterization of the canonical quantization procedure.

Introduction

Consider the space P of complex-valued polynomials in two real variables. The Poisson bracket operation

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

makes P into a complex Lie algebra. The *Dirac problem* asks if it is possible to derive from first principles a mapping θ from P to the algebra D of differential operators [with polynomial coefficients, acting on $L^2(\mathbb{R})$] which produces the correct spin zero quantization of classical mechanical systems [2, 3]. For definiteness, we take this to mean that θ should transform each function on $\mathbb{R}^2$ in the form of a Hamiltonian of a classical system

$$h(x,y) = \tfrac{1}{2}y^2 + V(x),$$

where V is a polynomial, into the Schrödinger operator on $L^2(\mathbb{R})$,

$$H = -\tfrac{1}{2}\Delta + V.$$

Here θ should certainly be a linear mapping, which hopefully would transform Poisson brackets into commutator brackets in the following sense:

$$\theta(f)\theta(g) - \theta(g)\theta(f) = \sqrt{-1}\,\theta(\{f,g\}). \tag{0.1}$$

* This research was supported by a senior fellowship of the Science Research Council, U.K., at the University of Newcastle, and by National Science Foundation grant MCS80-06264