Commun. Math. Phys. 88, 357–385 (1983)

Communications in Mathematical Physics

The Heat Equation with Singular Coefficients

I. Operators of the Form $-\dfrac{d^2}{dx^2}+\dfrac{\kappa}{x^2}$ in Dimension 1*

Constantine J. Callias

Department of Mathematics, Columbia University, New York, NY 10027, USA

Abstract. The small time asymptotics of the kernel of e^{-tH} is defined and derived for $H=\dfrac{d^2}{dx^2}+\dfrac{\kappa}{x^2}$ on $\mathbb{R}^1$. Lemmas on singular asymptotics in the sense of distributions are formulated and used. The results are applied to derive an index formula on $\mathbb{R}^1$.

1. Introduction

In this and a few subsequent papers I intend to discuss the heat equation with coefficients containing singularities of the kind exemplified by $\dfrac{\kappa}{x^2}$. I shall study some general properties and asymptotic behavior with an eye toward applications in spectral theory and in quantum theory. From the point of view of the latter, one only needs to concentrate on Euclidean space, but our discussion will eventually be extended to arbitrary manifolds in a straightforward way.

My motivation for this study originated in an earlier attempt to compute quantum corrections to classical solutions in Euclidean Yang-Mills theory [1]. In a steepest descent approximation scheme one has to calculate the determinant of a second order expansion operator about the extremum of the exponential in a function space integral. The operator in question is a second order differential operator on $\mathbb{R}^4$ and the determinant is defined by the derivative of the analytic continuation of the zeta function

$$\zeta(\lambda)=\frac{1}{\Gamma(\lambda)}\int_0^\infty ds\, s^{\lambda-1}\,\mathrm{Tr}(e^{-sH}-e^{-sH_0}) \tag{1.1}$$

to $\lambda=0$ according to the formula

$$\ln\det H=-\zeta'(0). \tag{1.2}$$

* Work partly supported by the National Science Foundation Grant No. 80-08894 MCS