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Gauge Potentials and Bundles Over the 4-Torus

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Abstract. The construction of principal bundles over a four dimensional torus is considered. The class of groups considered is $SU(n)/Z_n$, and for this class the Pontrjagin class has even integer values.

1. Introduction

This paper considers principal fibre bundles over a four-dimensional torus. Physically a four-dimensional torus corresponds to space-time being a kind of Euclidean box with periodic boundary conditions. Fibre bundles enter when one considers non-Abelian gauge fields inside this box. This physical picture has been considered by a number of people, cf., for example, [1] and references cited therein.

In [1] it is argued that the gauge groups $SU(n)/Z_n$, $n = 2, 3, \ldots$ are physically important (Z_n stands for the centre of the group $SU(n)$, hence for each n, Z_n is isomorphic to the n^{th} roots of unity). The topology of space-time is $S^1 \times S^1 \times S^1 \times S^1$, where S^1 is a unit circle. We shall denote space-time by T^4. Underlying the non-Abelian gauge field is a fibre bundle and so we are led to the construction of all $SU(n)/Z_n$ bundles over T^4. We describe, in what follows, a method for carrying out this construction. In Sect. 2 we treat the case $n = 2$, and in Sect. 3 the case $n > 2$. An important mathematical tool in the calculations will be the generalised cohomology theory known as K-theory.

2. The $n = 2$ Case

When $n = 2$ there is the well known result, of a kind typical for Lie groups of low dimension, that $SU(2)/Z_2 \simeq SO(3)$. Thus we wish to construct all SO(3)-bundles over T^4. In contrast to the case where the base space is a sphere S^k the calculation is not completely straightforward. It turns out to be most easily accomplished by resorting to a well known mathematical tool of bundle theory known as K-theory. K-theory is a kind of generalised cohomology theory defined for vector bundles. For an introduction to K-theory, cf. the works cited in [2]. The K-theory for T^4 considers all vector bundles E over T^4 and assembles them together into equivalence classes—two bundles E and F are equivalent if the addition of a trivial