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On the Schrödinger Equation and the Eigenvalue Problem

Peter Li[1,*] and Shing-Tung Yau[2]

1 Department of Mathematics, Stanford University, Stanford, CA 94305, USA
2 School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract. If λ_k is the k^{th} eigenvalue for the Dirichlet boundary problem on a bounded domain in $\mathbb{R}^n$, H. Weyl's asymptotic formula asserts that $\lambda_k \sim C_n\left(\frac{k}{V(D)}\right)^{2/n}$, hence $\sum_{i=1}^{k} \lambda_i \sim \frac{nC_n}{n+2} k^{\frac{n+2}{n}} V(D)^{-2/n}$. We prove that for any domain and for all k, $\sum_{i=1}^{k} \lambda_i \geqq \frac{nC_n}{n+2} k^{\frac{n+2}{n}} V(D)^{-2/n}$. A simple proof for the upper bound of the number of eigenvalues less than or equal to $-\alpha$ for the operator $\Delta - V(x)$ defined on $\mathbb{R}^n$ $(n \geqq 3)$ in terms of $\int_{\mathbb{R}^n} (V+\alpha)_-^{n/2}\, dx$ is also provided.

0. Introduction

In this paper, we study the eigenvalue problem with or without potential. We mainly concern ourself with bounded domains in $\mathbb{R}^n$ for the case of the Laplace operator. If D is a bounded domain in $\mathbb{R}^n$ we consider the eigenvalue problem

$$\Delta\phi = -\lambda\phi, \quad \text{on} \quad D$$

and

$$\phi|_{\partial D} \equiv 0. \tag{$*$}$$

The discreteness of the spectrum of Δ allows one to order the eigenvalues $(0<)\lambda_1 < \lambda_2 \leqq \ldots \leqq \lambda_k \leqq \ldots$, monotonically.

In the case of the Schrödinger equation, we consider potentials whose negative part are in $L^{n/2}(\mathbb{R}^n)$. If $V(x)$ is a potential function defined on $\mathbb{R}^n$ for $n \geqq 3$ and suppose $\int_{\mathbb{R}^n} V_-(x)\, dx$ is finite (see Sect. 2 for definition), it is then well known that the operator $\Delta - V(x)$ has discrete spectrum on the negative real line, i.e., the number of non-positive eigenvalues $N(0)$ for the problem

$$(\Delta - V(x))\,\phi(x) = -\mu\phi(x), \quad \text{on} \quad \mathbb{R}^n \tag{$**$}$$

is finite.

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