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Computing the Topological Entropy of Maps

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Abstract. We give an algorithm for determining the topological entropy of a unimodal map of the interval given its kneading sequence. We also show that this algorithm converges exponentially in the number of letters of the kneading sequence.

It is by now well known that iterated maps of an interval, when viewed as dynamical systems, account for some of the irregular behaviour observed in physics. There are three commonly used indicators for the complexity of such systems: The metric entropy, the Liapunov exponent, and the topological entropy. Here we shall discuss an efficient method for calculating the weakest of these notions, namely the topological entropy.

A possible way of defining the topological entropy $h(f)$ of a function f is given by

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log_2 N(f^n), \tag{1}$$

where f^n denotes the n^{th} iterate of f, and $N(g)$ is the number of monotone pieces of the graph of the function g. Thus the topological entropy, if positive, measures the exponential growth rate of the number of laps of f^n as n increases. If f is continuous and has a single extremum, then $h(f)$ takes values in $[0, 1]$. If $h(f)$ is positive, then the map f has complex behaviour in the following sense:

1) f has infinitely many different types of aperiodic and periodic orbits. In particular, even if f has a stable periodic orbit, complicated transient behaviour will be observed.

2) Although the topological entropy gives essentially no information about attractors, it indicates, when positive, a sensitivity of the dynamical system to external noise [2, 4].

In this note, we prove that $h(f)$ can be computed efficiently from the orbit of the critical point of f, using the so-called kneading determinant of Milnor and Thurston [5].[1] Our theorem below shows that $h(f)$ can be computed with an error

1 See also [1] for background material