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Pure Point Spectrum for Discrete Almost Periodic Schrödinger Operators

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Abstract. The finite difference Schrödinger operator on $\mathbb{Z}^m$ is considered

$$Hu_j = \left(\sum_{\nu=1}^{m} D_\nu^2\right) u_j + \frac{1}{\varepsilon} q_j u_j, \qquad u \in \ell^2(\mathbb{Z}^m),$$

where $\sum_{\nu=1}^{m} D_\nu^2$ is the difference Laplacian in m dimensions. For ε sufficiently small almost periodic potentials q_j are constructed such that the operator H has only pure point spectrum. The method is an inverse spectral procedure, which is a modification of the Kolmogorov-Arnol'd-Moser technique.

1. Introduction

There has been recent interest in the nature of the spectrum of the Schrödinger operator endowed with an almost periodic potential. In contrast to the periodic case, in which there is the classical band structure and the spectrum is all absolutely continuous, there is a wide range of other possibilities. For example the spectrum could be nowhere dense, [1, 12], and pure point or singular continuous spectrum could occur [2]. Somewhat more is known about the spectrum of finite difference Schrödinger operators on $\ell^2(\mathbb{Z})$, [4, 17], especially in the "almost Mathieu" case, in which the potential is given by a pure cosine with period incommensurate with the lattice period. However the existence of pure point spectrum, that is, of $\ell^2(\mathbb{Z})$ eigenvectors, has only been demonstrated in several special cases, for example [4, 16]. In this paper I construct, via an inverse spectral procedure, finite difference Schrödinger operators

$$\begin{aligned} (Hu)_j &= \sum_{|k|=1} u_{j+k} + \lambda q_j u_j, \qquad u_j \in \ell^2(\mathbb{Z}^m), \qquad j \in \mathbb{Z}^m, \\ |j| &= \sum_{\nu=1}^{m} |j_\nu|, \end{aligned} \tag{1.1}$$