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Nonlinear Schrödinger Equations and Sharp Interpolation Estimates

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Abstract. A sharp sufficient condition for global existence is obtained for the nonlinear Schrödinger equation

(NLS) $$2i\phi_t + \Delta\phi + |\phi|^{2\sigma}\phi = 0, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}^+,$$

in the case $\sigma = 2/N$. This condition is in terms of an exact stationary solution (nonlinear ground state) of (NLS). It is derived by solving a variational problem to obtain the "best constant" for classical interpolation estimates of Nirenberg and Gagliardo.

I. Introduction

The "best constant" of an interpolation estimate among various norms often has an analytical or geometrical significance [2, 23].

The main objective of this paper is to present a relationship between the best constant for a classical interpolation inequality due to Nirenberg and Gagliardo, and a sharp criterion for the existence of global solutions to the nonlinear Schrödinger equation:

$$2i\frac{\partial\phi}{\partial t} + \Delta\phi + |\phi|^{2\sigma}\phi = 0, \quad \phi = \phi(x,t), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}^+, \quad \phi(x,0) = \phi_0(x). \tag{I.1}$$

in the critical case $\sigma = 2/N$.

We will use the notation $\|f\|_p \equiv \left(\int_{\mathbb{R}^N} |f(x)|^p dx\right)^{1/p}$.

In Sect. II we determine the best constant $C_{\sigma,N}$ for the interpolation estimate [12, 13, 22]:

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leqq C_{\sigma,N}^{2\sigma+2} \|\nabla f\|_2^{\sigma N} \|f\|_2^{2+\sigma(2-N)}, \quad \text{if} \quad 0 < \sigma < \frac{2}{N-2}, \quad N \geqq 2. \tag{I.2}$$

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