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Spacelike Hypersurfaces with Prescribed Boundary Values and Mean Curvature

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Abstract. We consider the boundary-value problem for the mean curvature operator in Minkowski space, and give necessary and sufficient conditions for the existence of smooth strictly spacelike solutions. Our main results hold for non-constant mean curvature, and make no assumptions about the smoothness of the boundary or boundary data.

Introduction

An important problem in classical Relativity is that of determining existence and regularity properties of maximal and constant mean curvature hypersurfaces. These are spacelike submanifolds of codimension one in the spacetime manifold, with the property that the trace of the extrinsic curvature is respectively zero, constant. Such surfaces are important because they provide Riemannian submanifolds with properties which reflect those of the spacetime. For example, if the weak energy condition is satisfied, then a maximal hypersurface has positive scalar curvature. This fact was important in the initial proof of the positive mass conjecture [SY]. Other applications can be found in [ES] and [MT].

However, the use of these surfaces is restricted presently because their analytical properties are not well understood. By considering extrema of the associated variational problem, natural conditions for the existence of weak solutions can readily be established [Av]. These extrema are *a priori* only Lipschitz-continuous and may be lightlike ("go null", in the terminology of [MT]). Smoothness will follow from non-linear elliptic theory, provided we can show that they do not go null.

We consider this problem in flat Minkowski space $\mathbb{L}^{n+1}$, and give necessary and sufficient conditions to ensure that extrema of the variational problem do not go null. To be precise, we show (in Theorem 4.1) that for a given bounded domain $\Omega \subset \mathbb{R}^n$ there is a smooth strictly spacelike solution of the Lorentz mean curvature equation with specified boundary values on $\partial\Omega$, provided only that the given boundary data spans some spacelike hypersurface, and the mean curvature function is smooth and bounded on $\Omega \times \mathbb{R}$. Since this result makes no assumptions about the regularity of the boundary data or boundary, it can be applied to problems over unbounded domains. We illustrate this in Sect. 4, using a barrier construction due to Treibergs [T].

We mention that some results for the Dirichlet problem for zero mean curvature were obtained in [B, F]; however both these authors make very

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