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A Short Proof of a Kupershmidt-Wilson Theorem

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Abstract. This is a short elementary proof of a statement originally observed by Adler, then pursued by the author, by Kupershmidt and Wilson, and in a more general setting by Drinfeld and Sokolov.

Gardner, Zakharov, and Faddeev have found a Hamiltonian structure for the Korteweg-de Vries equation. Gelfand and the author constructed much more general Hamiltonian structures and integrable equations which were Hamiltonian in those structures. Adler [1] supposed that there is also a second Hamiltonian structure for those equations and gave without proof an expression for this second symplectic form. This conjecture was confirmed by Gelfand and the author [2]. Kupershmidt and Wilson [3] showed that "the second symplectic form" is equivalent to a very simple symplectic form of the Gardner-Zakharov-Miura type and this equivalence is given by a "general Miura transformation" introduced by these authors. This also follows from a recent paper of Drinfeld and Sokolov [4].

Since the theorem of Kupershmidt and Wilson is very important we think that it is useful to give a very simple proof of this theorem by a direct calculation.

We remind the reader the formulation of this theorem. Let R be a ring of formal operators $\sum_{-\infty}^{m} a_i\partial^i$ where a_i belong to a differential algebra $\mathcal{a}$ and ∂^i are symbols; the multiplication is defined by the rules

$$\partial^i\partial^j=\partial^{i+j}, \qquad \partial^i f = f\partial^i + \binom{i}{1} f'\partial^{i-1} + \binom{i}{2} f''\partial^{i-2} + \dots, \qquad f\in \mathcal{a}\,.$$

Let R_+ be a subring of "differential operators" $\sum_0^m a_i\partial^i$, and let R_- be a subring of "Volterra's integral operators," $\sum_{-\infty}^{-1} a_i\partial^i$. We denote

$$\left(\sum_{-\infty}^{m} a_i\partial^i\right)_+ = \sum_0^m a_i\partial^i, \qquad \left(\sum_{-\infty}^{m} a_i\partial^i\right)_- = \sum_{-\infty}^{-1} a_i\partial^i, \qquad \operatorname{res} \sum_{-\infty}^{m} a_i\partial^i = a_{-1}\,.$$

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