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More Surprises in the General Theory of Lattice Systems

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Abstract. I use Israel's methods to prove new theorems of "ubiquitous pathology" for classical and quantum lattice systems. The main result is the following: Let Φ be any interaction and ϱ be any translation-invariant equilibrium state for Φ (extremal or not). Then there exists a sequence $\{\Phi_k\}$ of interactions converging to Φ, having extremal (or even unique) translation-invariant equilibrium states ϱ_k, such that $\{\varrho_k\}$ converges to ϱ. In certain situations the perturbations $\Phi_k - \Phi$ can be chosen to lie in a cone of "antiferromagnetic pair interactions." I discuss the connection with results of Daniëls and van Enter, and point out an application to the one-dimensional ferromagnetic Ising model with $1/r^2$ interaction (Thouless effect).

1. Introduction

Israel [1, 2] has recently introduced elegant abstract methods for the study of classical or quatntum lattice systems in statistical mechanics with general translation-invariant interaction. Two of his results are quite surprising, for they assert that situations generally considered to be "pathological" are in fact ubiquitous:

(a) Let $\varrho_1, \ldots, \varrho_n$ be *any* finite family of ergodic translation-invariant states with finite mean entropy. Then there exists some interaction Φ (in a certain Banach space $\mathscr{B}$ of interactions) for which *all* these states are equilibrium states.

(b) There is a *dense* set of interactions in $\mathscr{B}$ each of which has *uncountably*[1] many ergodic equilibrium states[2] (i.e. uncountably many pure phases)!

1 In fact, the cardinality is exactly that of the continuum. This is because the extreme points of a metrizable compact convex set are a G_δ [2, Lemma IV.3.1], hence a Borel set; and it can be shown, *without* invoking the continuum hypothesis, that every uncountable Borel (or even analytic) set in a complete separable metric space has cardinality exactly that of the continuum [3]

2 The proof of Lemma V.2.3 in [2] is incomplete: it needs the additional remark that the set $\{\varrho : F(\varrho) \equiv P(\Phi_0) + \varrho(A_{\Phi_0}) - s(\varrho) < \delta\}$ is dense in the set $\{\varrho : F(\varrho) \leqq \delta\}$. To see this, assume that $F(\varrho_0) = \delta$; then taking some ϱ_1 such that $F(\varrho_1) = 0$ [i.e., ϱ_1 is an equilibrium state for Φ_0] and letting $\varrho_t = (1-t)\varrho_0 + t\varrho_1$, we have $\varrho_t \to \varrho_0$ as $t \to 0$ and $F(\varrho_t) < \delta$ for $t > 0$ by the convexity (actually affineness) of F. I thank Professor Israel for supplying this observation in response to my query, and for giving me permission to include it here

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