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On the Positivity of the Effective Action in a Theory of Random Surfaces

E. Onofri*

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract. It is shown that the functional $S[\eta]=\frac{1}{24\pi}\int(\frac{1}{2}|\nabla\eta|^2+2\eta)d\mu_0$, defined on C^∞ functions on the two-dimensional sphere, satisfies the inequality $S[\eta]\geqq 0$ if η is subject to the constraint $\int(e^\eta-1)d\mu_0=0$. The minimum $S[\eta]=0$ is attained at the solutions of the Euler–Lagrange equations. The proof is based on a sharper version of Moser–Trudinger's inequality (due to Aubin) which holds under the additional constraint $\int e^\eta \mathbf{x}\, d\mu_0=0$; this condition can always be satisfied by exploiting the invariance of $S[\eta]$ under the conformal transformations of S^2. The result is relevant for a recently proposed formulation of a theory of random surfaces.

1. Introduction

Let $ds^2=e^\eta ds_0^2$ denote a Riemannian metric on the two-dimensional sphere S^2, conformal to the standard metric $ds_0^2=d\theta^2+\sin^2\theta d\phi^2$. The points of S^2 will be parametrized, as usual, by a unit vector $\mathbf{x}$, by polar co-ordinates (θ,ϕ) or by a complex variable ξ, related to $\mathbf{x}$ by stereographic projection, i.e., $\xi=\cot\frac{\theta}{2}e^{i\phi}=(x_1+ix_2/1-x_3)$. The conformal factor e^η is assumed to be C^∞. Let $\Delta=e^{-\eta}\Delta_0$ be the Laplace–Beltrami operator associated to ds^2 and let $0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\ldots\leqq\lambda_n\leqq\ldots\to\infty$ be the spectrum of $-\Delta$ (Δ_0 and $\{\lambda_n^0\}$ will denote the corresponding objects belonging to ds_0^2).

It was shown in Ref. [1] that the limit

$$\frac{\det\Delta}{\det\Delta_0}\equiv\lim_{n\to\infty}\prod_{k=1}^{n}\frac{\lambda_k}{\lambda_k^0}=e^{-S[\eta]} \tag{1}$$

exists provided that e^η is normalized, i.e.,

$$\int(e^\eta-1)d\mu_0=0, \tag{2}$$

* On leave from: Istituto di Fisica dell'Università di Parma, Sezione di Fisica Teorica, Parma, Italy

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