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Gauge Theories on Four Dimensional Riemannian Manifolds

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Abstract. This paper develops the Riemannian geometry of classical gauge theories – Yang-Mills fields coupled with scalar and spinor fields – on compact four-dimensional manifolds. Some important properties of these fields are derived from elliptic theory: regularity, an "energy gap theorem", the manifold structure of the configuration space, and a bound for the supremum of the field in terms of the energy. It is then shown that finite energy solutions of the coupled field equations cannot have isolated singularities (this extends a theorem of K. Uhlenbeck).

Introduction

One of the major discoveries of physics in this century is the recognition that non-abelian Lie groups play a role in particle physics. For many years this was regarded as a peculiar aspect of quantum mechanics having no classical analogue. Then in 1954 C. N. Yang and R. Mills proposed a classical field theory incorporating these groups. Recently their theory has received considerable attention from both mathematicians and physicists.

Yang-Mills theory is easily described in terms of modern differential geometry. One begins with a principal bundle P with compact Lie structure group G over a manifold M. The Yang-Mills field is then the curvature Ω of a connection ∇ on P which is a critical point of the action

$$A(\nabla)=\int_M |\Omega|^2 .$$

When G is the circle group the Yang-Mills field satisfies Maxwell's equations.

Physically, Yang-Mills fields represent forces. As such they interact with a second type of field – the field of a particle. This is interpreted as a section ϕ of a vector bundle associated to P and the action for the system is essentially

$$A(\nabla,\phi)=\int_M |\Omega|^2+|\nabla\phi|^2-m^2|\phi|^2 ,$$

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